Targeting in chaotic scattering

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We consider a Hamiltonian system model with an orbit that is stabilized on an unstable periodic orbit embedded in an unstable chaotic set. We then attempt by means of a small control to target a position outside the original chaotic invariant set. This work illustrates how this can be accomplished using the example of the chaotic scattering set resulting from billiard-type motion in the presence of three hard circular disks. [S1063-651X(98)08605-X]

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I. INTRODUCTION

A. Targeting

The inherent exponential sensitivity of chaotic time evolutions to perturbations is the hallmark of chaotic systems. This sensitivity can be exploited in direct trajectories to some desired final state by the use of a carefully chosen sequence of small perturbations to some control parameters [1]. These perturbations can be so small that they do not significantly change the system dynamics, but enable the intrinsic system dynamics to drive the trajectory to the desired final state. This process has been called *targeting*. Targeting in chaotic dynamical systems has received much attention in recent years [2–9]. Since in chaotic systems small perturbations eventually produce large effects, the objective is to find methods that allow one to decide when and how judiciously chosen perturbations should be applied in order to achieve the desired effect.

In this area, one significant result was achieved with the spacecraft International Cometary Explorer (ICE) [10]. The spacecraft International Sun-Earth Explorer 3 (ISEE-3) was launched on August 12, 1978, with the purpose of investigating the solar-terrestrial relationship at the outermost boundaries of the Earth's magnetosphere. It was parked in an elliptical halo orbit about the Lagrange libration point L1, where it continuously monitored changes in the near-Earth interplanetary medium. (Dynamically, L1 is an unstable fixed point in the Earth-moon system in a rotating frame in which the Earth and moon are stationary.) In 1982 it was suggested that this spacecraft could be used to explore the comet Halley and the newly discovered Giacobini-Zinner comet, which were both then entering the solar system. However, there was only a relatively small amount of propellant in the spacecraft, and it was not initially clear that this would be sufficient. Nevertheless, a feasible orbit was eventually found. A maneuver was conducted on June 10, 1982, to remove the spacecraft from the parked orbit around the L1 point and place it in a transfer orbit involving a series of passages around the Earth and the moon. After fifteen small propulsive maneuvers and five lunar flybys, the spacecraft ISEE-3, then renamed ICE, was transferred to a heliocentric orbit on December 22, 1983. As planned, ICE traversed the tail of comet Giacobini-Zinner on September 11, 1986, and also approached the comet Halley in late March 1986, becoming the first spacecraft to directly investigate two comets [11]. We emphasize that the existence of the successful orbit is a result of the instability of the L1 parked location, and of the inherent instability caused by the chaotic character of the restricted three body problem: the spacecraft motion in the presence of the Earth and the moon.

In the present paper, motivated by the ICE achievement, we consider a paradigmatic situation of targeting. Our goal is to provide a targeting method that can be applied to a Hamiltonian system initially stabilized on an unstable periodic orbit embedded in a chaotic set. For our system, the chaotic set is a hyperbolic chaotic saddle, in which case we find that targeting can be very fast and efficiently achieved.

Since hyperbolic Hamiltonian chaotic sets have been extensively studied in the context of a chaotic scattering, our method will be shown in that context.

This article is organized as follows. In the next subsection we briefly review the concepts concerning chaotic scattering. In the subsequent subsection we situate our method in relation to other known methods. A model problem of a chaotic scattering is introduced in Sec. II. Our targeting method is described in Sec. III. The results from applying it to the proposed examples are presented in Sec. IV. A general discussion is given in Sec. V.

B. Chaotic scattering

As discussed below, chaotic scattering is characterized by ''sensitive dependence'' of output variables that characterize the particle trajectory after the scattering to small changes in an input variable that characterizes the trajectory before scattering [12]. This phenomenon has received much attention because many fundamental physical situations are of this type [3,13–17]. In general, scattering refers to a situation where the system motion is initially simple, then enters a scattering region where the motion can be more complicated, and then leaves the scattering region, again resuming simple motion [18].

We say that the scattering function (i.e., the output as a function of the input) is *singular* at a particular value of the input variable if any interval containing that input value pro-

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duces output variable values in a nonzero range that does not approach zero as the size of the input interval approaches zero. Thus two inputs that are *arbitrarily close* to a singular value can produce very different outputs. When the set of singular input variables is uncountable, and occurs on a Cantor set, we call the situation chaotic scattering, and we say that there is sensitive dependence of the output to small changes in the input.

The dynamics of scattering in a hyperbolic chaotic situation can be explained by the existence of a saddle chaotic invariant set [19], formed by the intersection of its stable and unstable manifolds, where the stable and unstable manifolds each consist of a Cantor set of roughly parallel surfaces. When a particle enters a scattering region close to the stable manifold, it stays near the saddle chaotic set for some time, and then escapes following a path close to the unstable manifold. The closer it initially is to the stable manifold, the more time it spends in the scattering region. If the initial condition of the particle is precisely on the stable manifold, the particle stays in the scattering region forever, and small deviations from this situation can lead to wild variations of the output [12,16,19–23].

One fundamental aspect of chaotic invariant sets is that they are typically permeated by an infinite dense set of unstable periodic orbits. Here we consider an orbit that is stabilized on one of these unstable periodic orbits. We use afterwards the term ''*parked*'' to describe this situation. In particular, we envision that the system has somehow been brought to the vicinity of the desired parking orbit, which is then stabilized by application of a small control, as in the method proposed by Lai, Tel, and Grebogi [7,8]. After being maintained in this unstable orbit, we assume that it becomes desirable to target some particular region of phase space. This scenario was, in fact, carried out, as we have already mentioned, by the ISEE-3 spacecraft in achieving its encounter with the comets Giacobini-Zinner and Halley. The reverse scenario, where we are interested in steering a trajectory from outside a scattering region to one of the previously mentioned unstable periodic orbits, will also be considered here.

These scenarios appear in many significant applications. In celestial mechanics, besides the guidance of a spacecraft nearby two heavy bodies that are moving in Keppler ellipses around their center of mass (restricted three body problem) [24,25], Petit and Hénon [17] investigated an interesting situation. They analyzed the case of two small bodies moving around a very heavy mass. When close encounters happen, a complicated motion can take place, in a situation that can be described by chaotic scattering. Consequently we can envisage the use of a targeting method to guide these satellites. Analogously, in particle accelerators, electromagnetic wave generators, and plasma physics, chaotic scattering happens in many situations as a result of the interaction between charge particles and electromagnetic fields [26,27]. After the interaction, a target method can be used to direct the trajectory of the particles to a specific region of the space. Also, chaotic scattering happens in many chemical reactions [28,29]. Even the simplest case of the reaction between two atoms bounded in a molecule with a third atom can be considered as a three body problem, where small differences in the interactions between them imply different results. Furthermore, the chemical reactions are usually made of a sequence of intermediate complexes of a finite average lifetime [30]. A target method can be used to accelerate the speed of reactions or stabilize intermediate results.

Next we address the issue of the applicability of previous developed target method to the situation of chaotic scattering.

C. Targeting in chaotic scattering

The idea of using the exponential sensitivity of a chaotic system to tiny perturbations to rapidly direct a system to a desired accessible state, i.e., targeting, was introduced by Shinbrot *et al.* [9]. In their work they described a method of targeting that conceptually can be applied to any chaotic system. This method is as follows: consider, in the phase space, a small region r_s around a source point s, and another small region r_t around the target point t. The objective is to find a point p_s interior to r_s that belongs to a trajectory in the phase space that goes from p_s to a point p_t, which is interior to r_t. To find p_s, the region r_s is iterated in the forward direction, while the region r_t is iterated in the backward direction until these iterated regions intersect each other in the phase space. When the intersection is found, the regions r_s and r_t are partitioned and the partitions, which implied, after the iterations, the intersection, are identified. This process is repeated with partitions progressively small, until the determination of p_s and p_t. The point p_s is then used to determine the value of the perturbation that must be applied to the system to direct it to p_s. As the system is in p_s, it will evolve following its own dynamics until it reaches p_t.

While this method works very well for dissipative systems, in general it is not suitable for chaotic scattering. Chaotic scattering is a situation of transient chaos, where the chaotic set is nonattractive, and, so, almost all initial conditions escape from the chaotic set except for a set of measure zero. To apply the Shinbrot method what is really iterated is a set of discrete points that represent the partition of the region around the source and the target points. However, as these points are iterated, almost all of them escape from the chaotic region. So, to find an eventual intersection of iterated regions, the points that represent the partition of the region around the source and targeting must be continuously redefined, to keep their iterations inside the scattering region. In fact, the accomplishment of this job requires an elaborate and computationally intensive procedure [31]. As the number of iterations is increased, for forward iterations, only points continuously close to the stable manifold will stay inside the scattering region; for backward iterations, only points close to the unstable manifold will stay inside the scattering region. Furthermore, suppose that this process was carried out and, as a result, the intersection of two iterated regions was found. Shinbrot's method applies a procedure of successive approximations to the solution by using a progressively small partition of the interval around source and target points. If the eventual solution is located very near the stable manifold, what will happen is that between any two points of the partition, no matter how close to each other they are, the system will present a very high sensitive dependence on initial conditions. This means that getting the solution point by using this process of successive approximations will be very difficult. The fact is that although this method can work in some situations, in general it will not work properly, besides the intensive calculation required to implement it.

For chaotic Hamiltonian systems, Lai *et al.* [7] proposed a method that can be used to stabilize trajectories in the neighborhood of some desired unstable periodic orbit (''control of chaos'') by using small perturbation. In subsequent work, Lai *et al.* [8] showed that the same method could be applied to a situation of chaotic scattering, even for nonhyperbolic chaotic scattering, where Kolmogorov-Arnold-Moser surfaces coexist with chaotic invariant sets. However, the authors stressed that the major problem about control of chaotic scattering was to bring the trajectories inside the region where their method of control could be applied (''controllable region''). This is because the chaotic set is nonattractive so there is just a finite probability that a trajectory that starts from a random chosen initial condition in the phase space gets inside the ''controllable'' region. Thus, the control of chaotic scattering can only be efficient if it operates connected with a targeting method.

In general, targeting in chaotic Hamiltonian systems is not easy to accomplish. Besides the coexistence of interwoven chaotic and quasiperiodic regions, the phase space is divided into layered components that are separated from each other by Cantori [32]. Typically, a trajectory initialized in one layer of the chaotic region wanders in that layer for a long period of time before it crosses the Cantori and wanders in the next layer. Bollt and Meiss proposed a method of targeting that can work even if source and target are located in different layers. They use the fact that long trajectories in a compact phase space are recurrent. Thus, they first identify a ''slow'' orbit that reaches the small region around the target. Then, in this orbit they identify all the recurrent ''loops.'' Using small perturbation, they try to find patches that skip the recurrent loops. This method, usually, significantly reduces the transport time, as they exemplify by using it to find a ''chaotic Earth-moon transfer orbit that requires 38% less total velocity boost than a comparable Hohman transfer orbit'' [5].

In a situation of chaotic scattering this method could be applied if finding a ''slow'' orbit that goes from the source point to the desired region around the target point were easy. In fact, finding such an orbit is very difficult and computationally intensive because the chaotic set is nonattractive [31], and almost all initial conditions escape from the scattering region, except a set of measure zero. For the same reason, it is even more difficult to find a long orbit that goes from the source to the target. Thus, this method would not be appropriate in chaotic scattering cases.

In this paper we introduce a targeting method that is significantly different from the ones previously mentioned. It was envisaged for the situation of hyperbolic chaotic scattering. The goal is to use a small perturbation to drive a trajectory from an unstable periodic orbit located inside the scattering region to a target point outside the scattering region. In fact, as the systems that we are interested in have time reversal symmetry, the trajectory that goes from the unstable periodic orbit to an outside point can be used in the backward direction. Thus, our method can be used in association with Lai's method of control of chaos to drive and stabilize

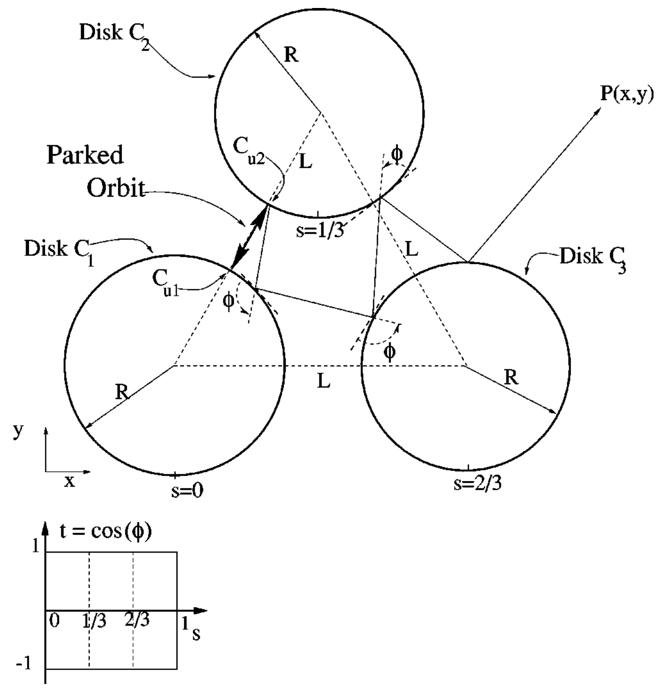

FIG. 1. Geometry of the problem. The disks are located at the vertices of an equilateral triangle with side L. P is the point to be targeted. s and ϕ are related to the surface of section that was defined for the problem.

in an unstable periodic orbit trajectories that come from outside the scattering region.

The method takes explicit advantage of the sensitive dependence on initial conditions. It finds, from an ensemble of trajectories that departs from the targeting point and heads to the scattering region, the trajectory that passes closest to the unstable periodic orbit. Then, by using time reversal symmetry, this trajectory is used as a reference trajectory in order to construct, by using small perturbation, a solution trajectory that departs from the unstable periodic orbit and goes to the target region.

In the following sections we introduce an example of the chaotic scattering set resulting from billiard type motion in the presence of three hard circular disks, describe our method, show how it can be applied to the proposed example, and present the results.

II. THE MODEL PROBLEM

We consider the two-dimensional billiard model that is schematically shown in Fig. 1. This system has been extensively studied [13,16,18,23,33], and consists of three circular hard disks, each of radius R, whose centers are located on the vertices of an equilateral triangle, of side length $L>2R$. Particles move in straight lines between perfectly elastic collisions with the disks, and with the angle of incidence equal to the angle of reflection at each collision.

We consider this system with the following parameters: $R=1.8$, $L=4.0$, and the circles located at the vertices of the triangle whose coordinates are $(0,0)$, $(2,2\sqrt{3})$, and $(4,0)$. Furthermore, we consider a particle initially bouncing between disks C_1 and C_2 along the unstable periodic orbit located on the segment of the line joining the centers of these circles, as shown in Fig. 1. We say that the particle is initially ''parked'' on this orbit. Our objective is to target, from this unstable orbit, a point P located outside the three-disk region. That is, by applying *small* perturbing controls we wish to direct the motion of the particle so that it hits the point P. For our example, we arbitrarily choose P to be at the position (8.0,4.0).

In what follows we find it convenient to introduce a surface of section that corresponds to the surfaces of the three disks. We parametrize the location of a point on the surface of section by an arclength variable s. The arclength variable s is measured in the counterclockwise sense, starting in each disk in the position corresponding to the point of the least y coordinate. It is defined such that the s interval $[0,1/3]$ belongs to the first disk, $[1/3,2/3]$ to the second, and $[2/3,1]$ to the third disk (see Fig. 1). (Thus, for example, s increasing from 1/3 to 2/3 takes a point initially located at the position labeled 1/3 in Fig. 1 and moves it counterclockwise around disk C_2 until it comes back to the starting point at $s=1/3$.) Furthermore, we also introduce $t=\cos\phi$, where ϕ is the angle between the incident direction at impact and the forward (counterclockwise) tangent.

Using this coordinate system, we will denote by f the map that transforms the point (s_n,t_n), associated with the nth collision of a particle with a disk, to the point (s_{n+1},t_{n+1}), associated with the next collision with a disk, i.e., f: $(s_n,t_n)\rightarrow(s_{n+1},t_{n+1})$. [The inset in Fig. 1 shows the (s,t) surface of section.] Note that for some (s,t) the subsequent orbit never again collides with a disk. In that case we say that $f(s,t)$ is undefined. When this occurs we think of the orbit as having escaped from the scattering region. Also note that there are discontinuities in the map at the positions $s=0,1/3,2/3,1$.

Poon *et al.* [33] have numerically estimated the stable and unstable manifold of this map and concluded that the invariant chaotic set is hyperbolic. Ding *et al.* [21,22] have gotten similar results in an analogous situation. We represent a trajectory that starts at a point A and after m subsequent bounces, hits a point B as a sequence of m points (s_1,t_1), $(s_2,t_2),\ldots(s_m,t_m)$ in the surface of section, where $t_i=\cos\phi_i$. Because of the time reversal symmetry, we have an associated ''inverse'' trajectory that starts at B and goes to A, defined also by m points (s_1',t_1'), $(s_2',t_2'),\ldots(s_m',t_m')$, where $t_i'=\cos\phi_i'$. These coordinates are related as follows: $s_1'=s_m$, $s_2'=s_{m-1},\ldots$, $s_m'=s_1$; $t_1'=-t_m$, $t_2'=-t_{m-1},\ldots$, $t_m'=-t_1$. The latter relation holds because the angle ϕ' in the backward direction is related to the angle ϕ at the same position s by $\phi'=\pi-\phi$, and thus $\cos(\phi')=-\cos(\phi)$.

III. TARGETING

Let C_{U1}, with coordinates (s_{u1},t_{u1}), where $t_{u1}=\cos\phi_{u1}=0$, be the point of the parked unstable periodic orbit located on the disk C_1, and let C_{U2}, with coordinates (s_{u2},t_{u2}), where $t_{u2}=\cos\phi_{u2}=0$, be the point of the parked unstable periodic orbit located on the disk C_2. Our aim is to find a ''small'' perturbation δ so that if this number is added to ϕ_{u1} or (exclusive) to ϕ_{u2}, the particle leaves the unstable periodic orbit, bounces $\tilde{N}_s$ times with the disks, and eventually hits the target point P. (Mechanically we can think of δ as resulting from a small impulsive thrust applied at the time of impact.)

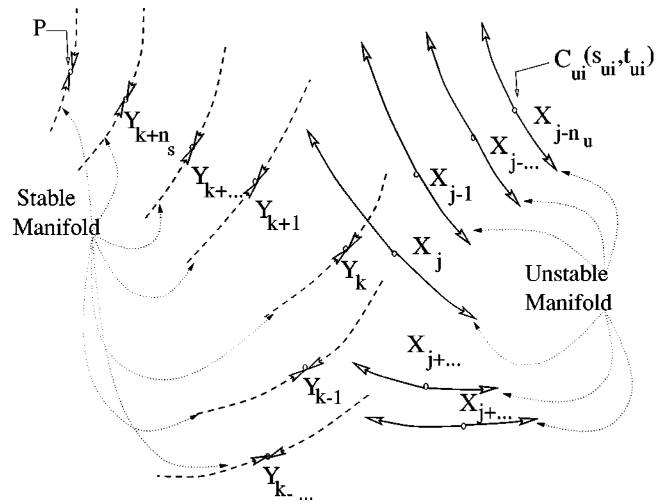

FIG. 2. Two trajectories that come close to each other.

Starting at P we randomly choose a large number of initial trajectory angles. We set a number N and retain those initial angles that lead to trajectories (s_i,t_i) that escape from the scattering region after N or more bounces with the disks. For N reasonably large the initial point (s_1,t_1) of such an orbit is located near the stable manifold [19]. From these trajectories, we select the one that comes closest to one of the surface of section parked orbit points C_{u1} or C_{u2}. Let $N_r\geqslant N$ denote the number of bounces of this selected trajectory with the disks, and (s_i,t_i) for $i=r_*\leqslant N_r$ denote the point of this trajectory in the surface of section that comes closest to the parked orbit. If we take this selected trajectory and obtain its *time inverse*, we have what we shall call the *reference trajectory*. That is, a trajectory that comes near the parked orbit, and escapes from the scattering region by following a path that passes through P. (Recall that to get the points of the time inverse trajectory in the surface of section we replace each t_i by $-t_i$, and reverse the time sequence of the points.)

The reference trajectory, represented in the surface of section, is a sequence of ordered points (s_i,t_i). If we imagine a particle following this trajectory, (s_{N_r},t_{N_r}) represents the last bounce of the particle with the disks before the particle escapes from the scattering region, following a trajectory that passes through P. The point of this trajectory that comes closest to the parked orbit in the surface of section is denoted (s_{r_m},t_{r_m}) with $r_m=N_r-r_*+1$. We denote by (s_{u_*},t_{u_*}), where $t_{u_*}=\cos\phi_{u_*}$, the point on the parked orbit that is nearest the point (s_{r_m},t_{r_m}) of the reference trajectory. [Thus (s_{u_*},t_{u_*}) is either C_{u1} or C_{u2}.]

The general situation of two orbits that come close to one another is depicted in Fig. 2, where an arbitrary surface of section was defined. There we can see the trajectories X_i and Y_i, which have the points X_j and Y_k as the respective position where the orbits come closest to each other.

For a hyperbolic situation, associated to each point on the invariant set there is a stable and an unstable manifold. We introduce at the position Y_{k+n_s} a small perturbation $\hat{\beta}e_\beta$ where e_β is a unit vector in the direction of the perturbation

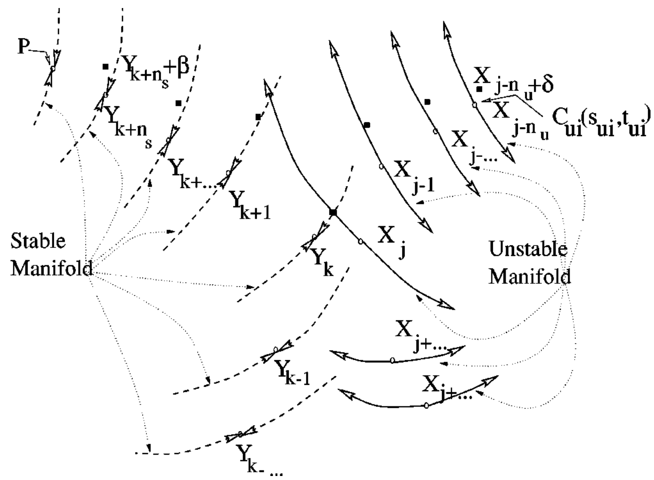

FIG. 3. Targeting method, solving for δ and β the following equation: $f^{n_u}(X_{j-n_u}+\delta e_\delta)=f^{-n_s}(Y_{k+n_s}+\beta e_\beta)$.

(see Fig. 3), and iterate this perturbed point n_s times backward in time. This will typically generate a nearby trajectory that will deviate progressively from the original trajectory at each backward iteration, expanding away from Y along the direction of the stable manifold at the points on the orbit Y [4,5]. (We assume that the direction of the small perturbation $\hat{\beta}e_\beta$ is not precisely such that it has no component in the stable direction.) We also introduce a small perturbation $\hat{\delta}e_\delta$ to the orbit X at the time $j-n_u$ where e_δ is a unit vector in the direction of the perturbation (see Fig. 3), and iterate this perturbed point forward in time n_u iterates. This will typically generate a nearby trajectory that will deviate progressively from the original trajectory at each forward iteration, expanding away from X along the direction of the unstable manifold at the points on the orbit X [4,5]. Consider that we can find values of the small perturbations $\hat{\delta}$ and $\hat{\beta}$ that solve the equation

$$f^{n_u}(X_{j-n_u}+\hat{\delta}e_\delta)=f^{-n_s}(Y_{k+n_s}+\hat{\beta}e_\beta). \tag{1}$$

This means that we have found a shadow trajectory that at time $j-n_u$ has a point that is $\hat{\delta}$ away from X_{j-n_u} and at n_s+n_u forward iterations in time is a $\hat{\beta}$ distance from the point Y_{k+n_s} of the trajectory Y. Thus the numbers n_u and n_s must satisfy $n_s \leqslant N_r-k$ and $n_u<j$. [Note that, since e_β is not necessarily aligned with the stable manifold at Y_{k+n_s}, forward iterates of $Y_{k+n_s}+\hat{\beta}e_\beta$ (if they exist) are expected to diverge from the trajectory Y.] We will initially be interested in the case where we associate the trajectory X with the parked orbit and the trajectory Y with a trajectory that goes through P, i.e., the reference trajectory. In that case, for small $\hat{\beta}$ and $N_r-(k+n_s)\geqslant 0$, a sufficiently small perturbation of the parked orbit to the point $X_{j-n_u}+\hat{\delta}e_\delta$ will yield an orbit that approaches the reference trajectory by following the direction of the stable manifold at the points of the reference trajectory, and thus comes close to the point P.

The application of this method to our problem is straightforward. Choosing n_u (the number of iterations in the forward direction), from the fact that the parked orbit is periodic of period two, and that (s_{u_*},t_{u_*}) is the position of the parked orbit in the surface of section that is closest to the reference orbit, we can find out if the perturbation $\hat{\delta}$ is to be applied at the point (s_{u1},t_{u1}) or (s_{u2},t_{u2}). Let us call this point (s_{u_p},t_{u_p}), where $t_{u_p}=\cos\phi_{u_p}$. In the same way, if (s_{r_m},t_{r_m}) is the point of the reference trajectory closest to (s_{u_*},t_{u_*}), the point where the perturbation $\hat{\beta}$ shall be applied is located n_s iterations away in the forward direction. We call this point (s_{r_p},t_{r_p}), where $t_{r_p}=\cos\phi_{r_p}$.

As previously stated we take the initial perturbation of the parked orbit to be applied in the angle (as if it were gotten from an impulsive propellant maneuver). The equation to be solved is then

$$f^{n_u}[s_{u_p},\cos(\phi_{u_p}+\delta)]=f^{-n_s}[s_{r_p},\cos(\phi_{r_p}+\beta)], \tag{2}$$

where we have written the map function as $f(s,t)$. The values chosen for the parameters n_u and n_s have a direct effect on the order of magnitude of δ and β. In general, as seen in the next section, larger n_u and n_s lead to solutions of Eq. (2) with smaller values of the perturbations δ and β.

Equation (2) can be solved for δ and β by the Newton-secant method. Let us call (s_s,t_s) the point in the surface of section where the perturbed trajectories intersect each other. Therefore, if we consider a particle parked on the unstable periodic orbit, and if we apply a perturbation δ to the angle when the particle hits the circle in the position (s_{u_p},t_{u_p}), the particle will escape from the parked orbit, and will approach the direction of the stable manifold of the points at the reference trajectory by following a trajectory (that we call the *solution trajectory*), such that after n_u bounces with the disks, it will be at the point (s_s,t_s); after n_s more bounces the trajectory will be at the point $[s_{r_p},\cos(\phi_{r_p}+\beta)]$; for subsequent bounces, the particle will closely follow the reference trajectory, and so will escape from the scattering region passing close to the target point P.

We can predict the total number of bounces with the disks, previously defined as N_s, that the solution trajectory will have. The point (s_s,t_s) is in fact very close to the point in the surface of section where the parked orbit and the reference trajectory come closest to each other. After this point, the solution trajectory comes very close to the reference trajectory so that both of them have the same number of hits with the disks. By construction, one particle that follows the reference trajectory, starting at the point (s_i,t_i) for $i=r_m$, has N_r-r_m hits with the disks before leaving the scattering region and passes through the target point P. Therefore, the total number of hits with the disks of the solution trajectory N_s can be forecasted by the following equation:

$$N_s=n_u+(N_r-r_m). \tag{3}$$

In Eq. (3), n_u, as a parameter, has its value assigned before we start to follow the targeting procedure. However, the values of N_r and r_m are just known during the execution of the targeting procedure. This is so because these numbers are associated with a trajectory (the reference trajectory) that is picked as the one that comes closest to the parked orbit from a set of trajectories that start from P with randomly chosen departure angles. The only constraints that we know in advance for these parameters are that N_r can be any number equal to or greater than N, while r_m can be any number between 1 and N_r. So, by following our procedure we can target the point P, but we can not fulfill, in general, the requested goal of targeting the point P with a *desired* number $\tilde{N}_s$ of bounces with the disks.

However, we can change the procedure to carry out this goal: in the process of finding a reference trajectory from a set of trajectories that depart from P with a randomly chosen departure angle, we select the one that escapes from the scattering region after *exactly* N bounces with the disks and that comes closest *in its middle point* to one of the surface of section parked orbit point C_{u1} or C_{u2}. By restricting our choosing criteria, we impose a determination over the values of N_r and r_m.

Introducing these changes to our previous procedure, we have a *reference trajectory* that has $N_r = N$ bounces with the disks, and which the point (s_i, t_i) for $i = r_m = N/2 + 1$ or (exclusive) $i = r_m = N/2$ is the one of the trajectory that comes closest to the surface of section parked orbit points. Using these values of r_m in Eq. (3) we have

$$N_s = n_u + N/2 \quad \text{or} \quad n_u + N/2 - 1\,. \tag{4}$$

Equation (4) allows us to assign a proper value for N, for a fixed value of the parameter n_u, in order to imply a solution trajectory that has a specified number of bounces with the disks.

In the next section we will analyze the results that we obtained by using this method.

IV. RESULTS

As an example, we set the goal that we want to target the point P from the ''parked'' orbit $C_{u1} - C_{u2}$ previously specified by using a small perturbation δ such that the solution trajectory has at most 12 ($\tilde{N}_s = 12$) bounces with the disks.

In order to apply our targeting procedure we first need to assign values to N, and to the parameters n_u and n_s. Guided by Eq. (4) for $\tilde{N}_s = 12$, we decided to use $N = 15$ and $n_u = 5$. For the parameter n_s we used the value $n_s = 5$.

The next step is to find the reference trajectory, which is a trajectory that has N bounces with the disks, and which the point (s_i, t_i) for $i = r_m = N/2 + 1$ or (exclusive) $i = r_m = N/2$ is the one of the trajectories that comes closest to the surface of section parked orbit points. Using the method that we have described in the previous section, we got the scenario that can be seen in Fig. 4. This figure shows in the surface of section previously defined the middle point of the trajectories that depart from P, and have the desired number of N bounces inside the scattering region. In this figure the position of the parked orbit is also shown. In fact, this figure can be considered as an approximate representation of the invariant set [19].

Figure 5 shows the reference trajectory. We verify that it really passes very close to the parked orbit, which is located between the disks C_1 and C_2, on the line connecting their centers.

In Fig. 6 we have the evolution of the Newton-secant method in solving Eq. (2) for the problem. We have gotten

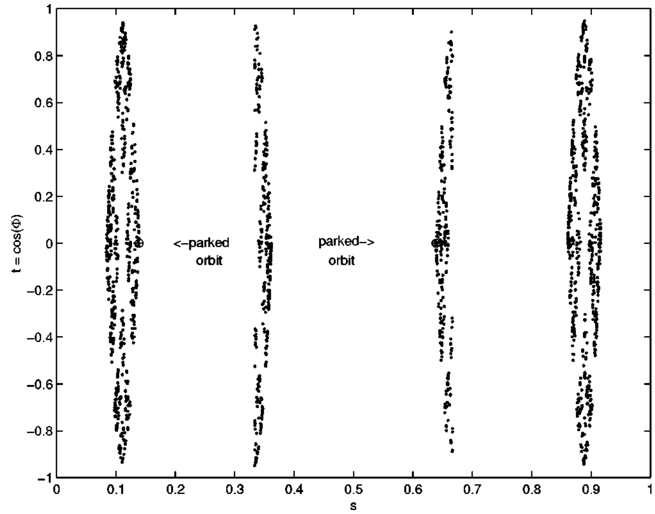

FIG. 4. Invariant set from P.

the following values for the perturbations: $\delta = 0.91869 \times 10^{-3}$ and $\beta = 0.57543 \times 10^{-3}$.

Applying this value of δ to the parked orbit, we have a trajectory that is shown in Fig. 7. The trajectory passes 0.385×10^{-2} away from the target, after 12 bounces with the disks, inside the scattering region. So, we fulfill our goal.

We can understand exactly how the method works by analyzing the ''error curve,'' i.e., the difference between the solution trajectory and the reference trajectory in their way through the scattering region. The initial point of this curve is taken immediately after the application of the perturbation δ. We consider this curve as it appears in our surface of section. So, the curve will be depicted as an ordered sequence of points (s_i, t_i), where each point will represent the difference between the solution trajectory and the reference trajectory in each hit with the disks.

The error curves appear in Figs. 8(a) and 8(b), for a specific case where $n_s = 5$, $n_u = 5$, and $N = 22$. In the former figure we have the error in the s coordinate, while in the latter we have the error in the t coordinate. In both curves, the abscissas represent the successive sequence of hits with the disks.

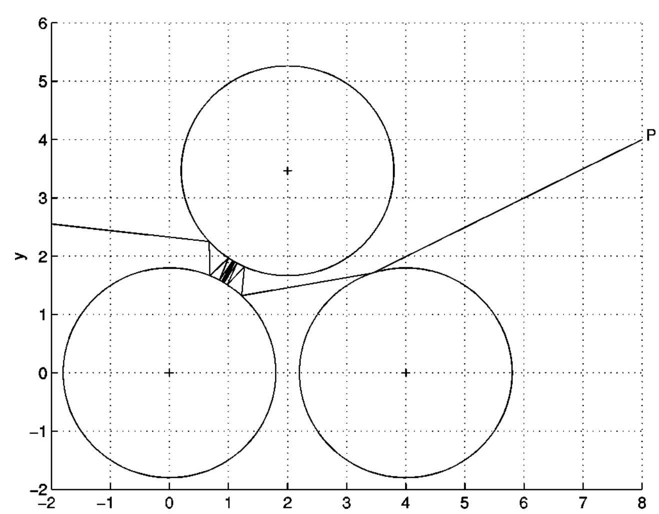

FIG. 5. Reference trajectory.

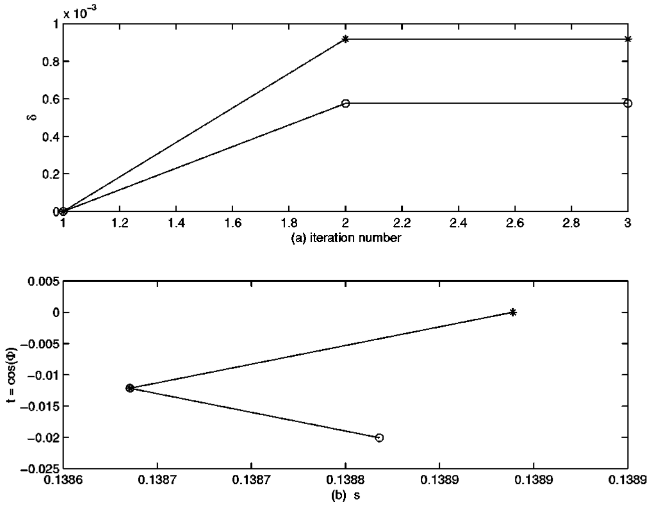

FIG. 6. Finding δ and β by using the Newton-secant method.

The shape of these curves is, in general, the same, independent of the values of n_u, n_s, and N: the difference between the solution trajectory and the reference trajectory decreases exponentially until a minimum point; after this point, the behavior changes, and the difference between those curves increases exponentially. So, we have a ''V-shaped'' curve. What happens is that, because of our targeting procedure, the solution trajectory comes exponentially close to the reference trajectory, following the stable manifold of the reference trajectory. The position of maximum proximity (minimum distance) is exactly the point of the reference trajectory used to calculate the β perturbation, according to Eq. (2). In Fig. 8, this point appears with abscissas $n_u+n_s=10$. After this point, we will have a situation of two trajectories that are initially close to one another. As the system is chaotic, the distance between the trajectories will increase exponentially, as expected.

This V-shaped curve is important because it suggests how we can choose the values of n_u and n_s for an usual problem where we want to target a point P through a trajectory that has no more than a specified number of bounces inside the scattering region: the closer the minimum of the V-shaped

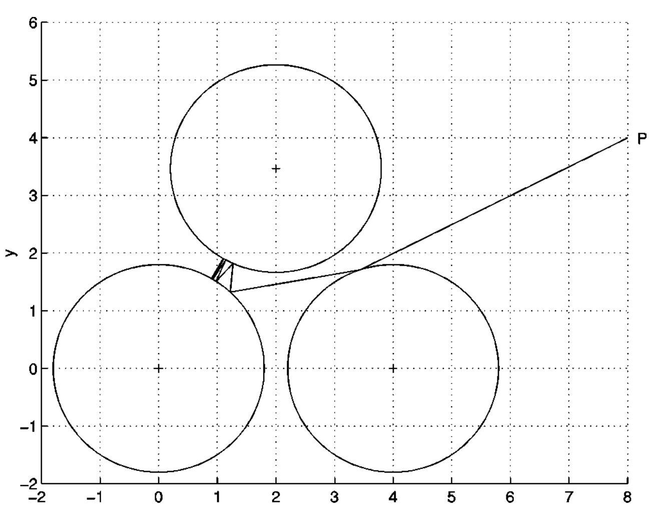

FIG. 7. Solution trajectory.

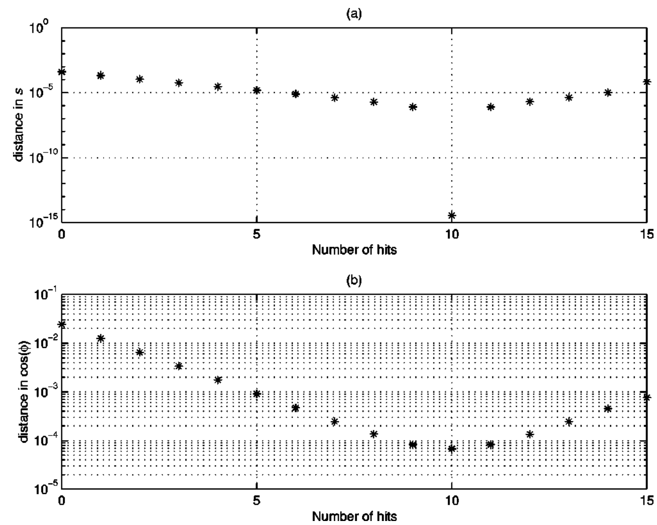

FIG. 8. V curve: error between the reference trajectory and the solution trajectory; (a) in s; (b) in t.

curve is of the last bounces with the disks, the closer the solution trajectory will be from the desired target point P. The position of the minimum can be shifted to the right by increasing the value of either n_u or n_s. However, the value n_s has a superior limit. Remembering Eq. (2), the β perturbation is to be applied in the surface of section n_s points after the position where the reference trajectory and the parked orbit come closest to each other. This point is located in the reference trajectory, as we have already said, $N/2$ or $N/2-1$ positions in the surface of section before the trajectory leaves the scattering region. So n_s cannot be greater than these values. In Fig. 9 we have a typical behavior of the distance from the solution trajectory to the target as n_s and n_u are increased. As expected, we have a decreasing curve.

In addition to implying trajectories that pass successively closer to the target point, increasing the values of n_u and n_s has another effect. This effect can be seen in the graphic that is showed in Fig. 10: the value of the perturbation δ that must be used to solve the problem decreases whenever n_u are increased. (The same behavior happens for β as a function of

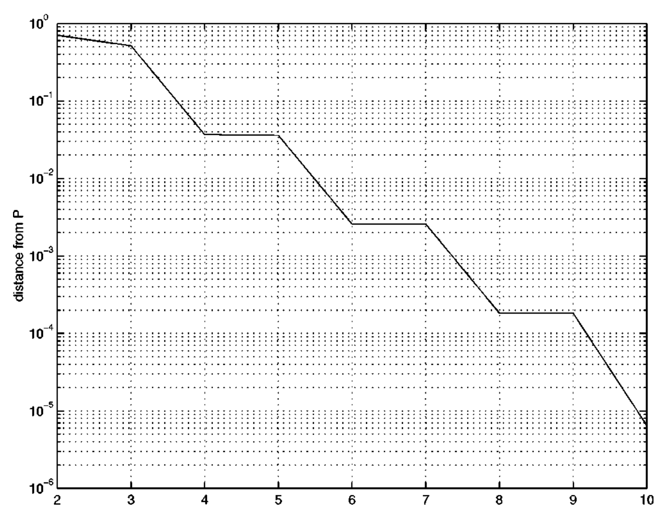

FIG. 9. The distance from the solution trajectory to the target point as the parameters n_s and n_u are changed.

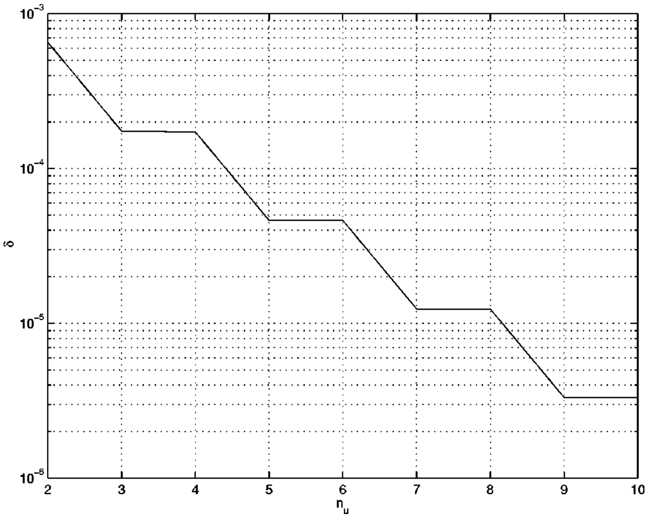

FIG. 10. Perturbation δ used to get the solution trajectory as the parameter n_u is changed.

n_s.) In fact, for large N, after some values of n_u typically we have for the solution of Eq. (2) values of δ such that the ratio between δ and the unperturbed angle is just two or three orders of magnitude greater than the machine accuracy.

We can overcome this problem by using our targeting method several times. This can be done in a very straightforward way by successively applying Eq. (2) to other points of the reference trajectory. Thus, consider a situation where N is large enough compared with n_s. Then, we apply our original method and find a δ_1 value. Using this δ_1 we will have a trajectory that leaves the parked orbit and approaches a minimum distance related to the reference trajectory. Let us call this solution trajectory 1. At any point of this trajectory we can apply again Eq. (2), considering that the δ_2 perturbation will be applied at this chosen point, which is m iterations away from the point where δ_1 was previously applied. In solving Eq. (2), we consider that the perturbation β_2 will be applied in a point of the reference trajectory that is also m iterations away from the point that was previously used to calculate β_1. Using this procedure, we have a solution trajectory 2. This procedure can be applied as many times as the length of the reference trajectory allows. The solution will be a union of partial solutions, which means that specific perturbation will need to be applied in specific points of the solution trajectory.

Figure 11 shows the result of the application of this procedure. There we compare the effect of applying the targeting procedure once, twice, and three times. In all cases the first δ is applied in accordance with our original targeting procedure, i.e., in one of the points of the unstable parked orbit. After the application of the first δ, other δ, when used, are applied, n_s positions spaced from each other in the solution trajectory as it appears in the surface of section. The overall effect of the utilization of several δ's is to move the minimum point of the V-shaped curve further away. It means that the solution trajectory continues to approach the reference trajectory, until the minimum point. Also, if we compare this minimum value with the one that happens when the target procedure is applied just once, the former is typically a few order of magnitude smaller than the latter.

In addition to increasing the value of the perturbations

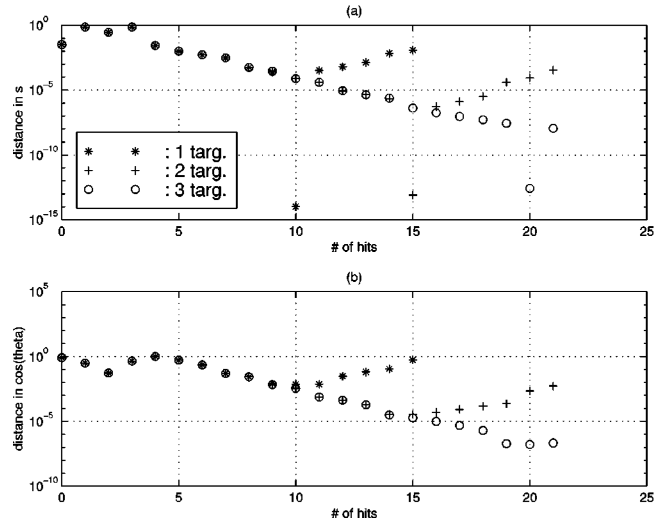

FIG. 11. V curve: error between the reference trajectory and the solution trajectory when the target algorithm is used once, two times, and three times; (a) in s; (b) in t.

that need to be applied to get the target point (as can be seen in Fig. 12), the use of our targeting method several times produces another important result. Consider a situation where there are nonideal effects as noise, state measurement error, and an imperfect determination of the system parameters. If we try to get the targeting by applying the target method just once, the influence of these nonideal effects will be to increase the value of the minimum of the V-shaped curve. As a consequence, the error between the solution trajectory and the reference trajectory can be so large that the solution trajectory passes far from the target point. By targeting several times, although the effect of increasing the minimum level of the V-shaped curve is still present, the procedure becomes strikingly more robust. This is because at each time the perturbation is applied we have a kind of trajectory correction that keeps the solution in the direction of the target. In Fig. 13 we have a typical curve, where we can see that the targeting procedure works well even for a noise of standard deviation equal to 10^{-5}. Note that this is a high

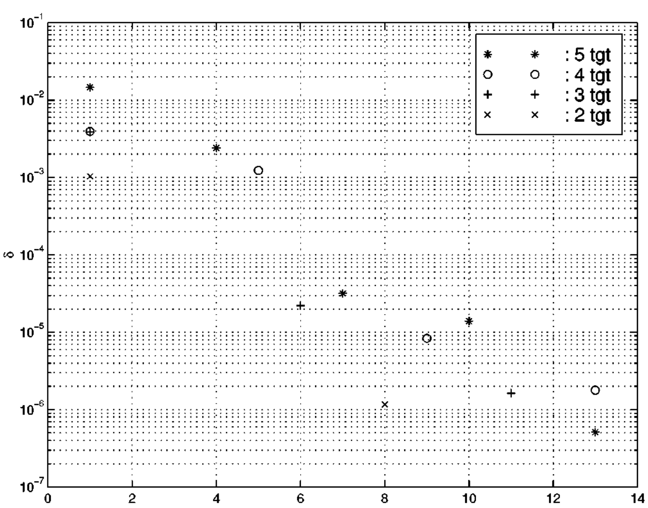

FIG. 12. Values of the perturbations δ's when the target algorithm is applied more than once.

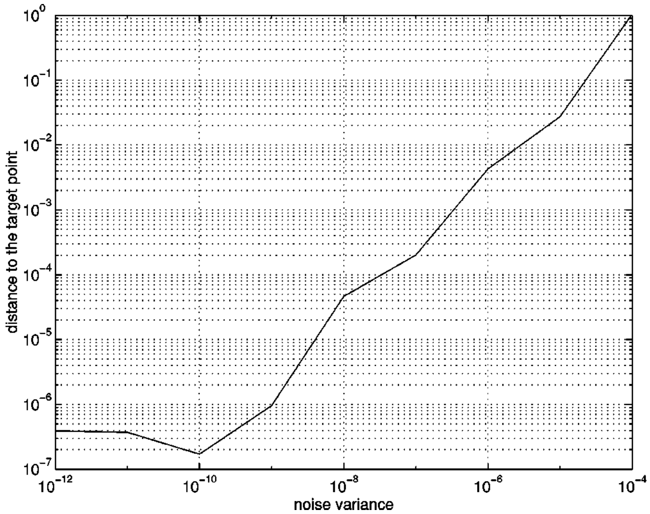

FIG. 13. The distance from the solution trajectory to the target point as the noise level is increased.

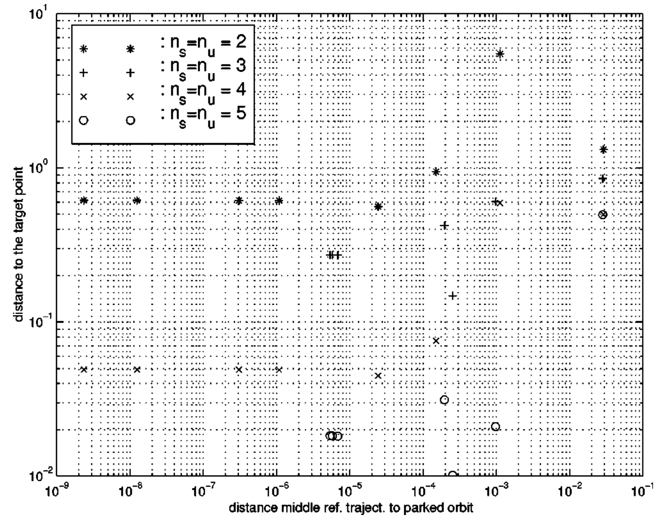

FIG. 14. Influence of the distance from the middle position of the reference trajectory to the parked orbit points over the distance from the solution trajectory to the target point.

level of noise for the case of chaotic scattering, where we have an extremely high sensitivity to small changes (the perturbations that are applied to steer the solution are typically of 10^{-3}).

The final issue that must be addressed is concerning the quantification of the distance in the surface of section between the middle point of the reference trajectory and the parked orbit so that our targeting method works. This question is relevant because the reference trajectory is found by using a set of random selected trajectories that depart from the target point P. Figure 14 shows how the distance from the middle point of the reference trajectory to the parked orbit points affects the distance from the solution trajectory to the target point P. From this figure, which is typical, we can see that after a well defined limit of proximity (about 10^{-4}), using reference trajectories closer to the parked orbit appears to have no significant effect in improving the proximity of the solution trajectory to the target point P.

V. CONCLUSION

We have presented a method that allows us to target a point outside a scattering region from any unstable periodic orbit located inside the scattering region. Despite the fact that we have illustrated the method by using an example where the parked orbit was one of period two, we can easily deal with any unstable periodic orbit without significant changes. In fact, we expect the method outlined here to be of general use for a class of targeting situations where it is desirable to start from a situation where the system is parked on an unstable periodic orbit embedded in a hyperbolic chaotic invariant set.

As general characteristics of our method we can say the following:

(i) The method is robust enough to deal with nonideal effects (noise, state measurement error, and an imperfect determination of the system parameters).

(ii) The method is flexible enough to be adapted to general situations involving orbits parked in an unstable orbit.

(iii) The perturbation that must be applied to get the target position can be adjusted inside a broad range of values.

Furthermore, the method also works in the reverse situation where we want to target an unstable periodic orbit located inside the chaotic scattering region departing from any point outside the scattering region. Thus, we can use it in association with the Lai method of control of chaos to capture in periodic orbits any particle sent in the direction of the scattering region. Besides, this association works even in a noisy environment, because the target method can be used to drive trajectories back to the stabilized orbit whenever a noise takes them out from the periodic orbit.

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